Approximating the area under the curve

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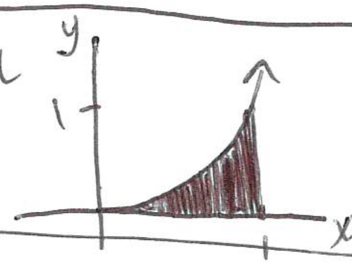
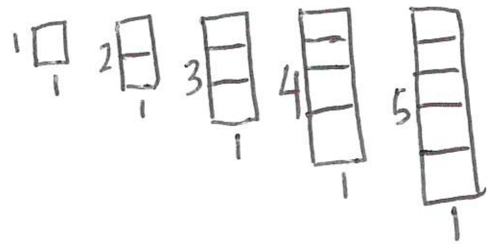
ES: How do we find the area under the curve?



Making a connection with summation

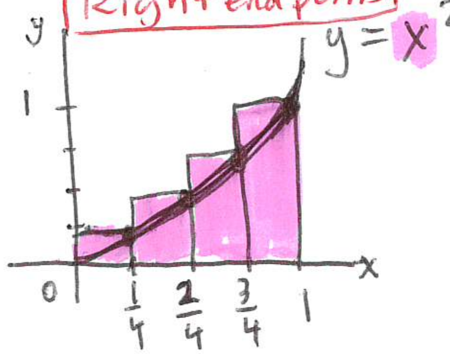


write the sum in sigma notation

$$= \sum_{i=1}^{5} 1i = \sum_{i=1}^{5} b \cdot h$$

Find the area under curve from 0 to 1 using rectangles

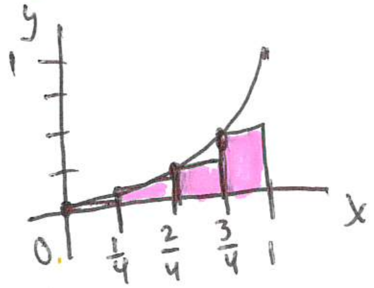
Right end points



Over approximation

number of rectangles	Area under the curve
4	$\sum_{i=1}^{4} bh = \sum_{i=1}^{4} \frac{1}{4}\left(\frac{1}{4}i\right)^2$ $= \frac{1}{4}\left(\frac{1}{4}\cdot 1\right)^2 + \frac{1}{4}\left(\frac{1}{4}\cdot 2\right)^2 + \ldots + \frac{1}{4}\left(\frac{1}{4}\cdot 3\right)^2 + \frac{1}{4}\left(\frac{1}{4}\cdot 4\right)^2$ $= 0.46875$
10	$\sum_{i=1}^{10} \frac{1}{10}\left(\frac{1}{10}i\right)^2 = 0.385$
100	$\sum_{i=1}^{100} \frac{1}{100}\left(\frac{1}{100}i\right)^2$ $= 0.33835$
1000	$\sum_{i=1}^{1000} \frac{1}{1000}\left(\frac{1}{1000}i\right)^2$ $= 0.33383$
∞	0.33333 $= \frac{1}{3}$

$\boxed{\text{Left end point}}$



$\boxed{\text{under approximation}}$

number of rectangles	Area under the curve
4	$\sum_{i=1}^{4} \frac{1}{4}\left(\frac{1}{4}(i-1)\right)^2$ $= \frac{1}{4}\left(\frac{1}{4}(0)\right)^2 + \frac{1}{4}\left(\frac{1}{4}(1)\right)^2 + \ldots$ $\ldots + \frac{1}{4}\left(\frac{1}{4}(2)\right)^2 + \frac{1}{4}\left(\frac{1}{4}(3)\right)^2$ $= 0.21875$
10	$\sum_{i=1}^{4} \frac{1}{10}\left(\frac{1}{10}(i-1)\right)^2$ $= 0.285$
100	$\sum_{i=1}^{100} \frac{1}{100}\left(\frac{1}{100}(i-1)\right)^2$ $= 0.32835$
1000	$\sum_{i=1}^{1000} \frac{1}{1000}\left(\frac{1}{1000}(i-1)\right)^2$ $= 0.33283$
∞	$0.33333 = \frac{1}{3}$

Finding Area using Limits	ES! How do we find the exact area under the curve?
Definition	If f is continuous and non negative on the Interval $[a,b]$, then the limit as $n \to \infty$ of both Lower and upper sums Exist and equal to each other:

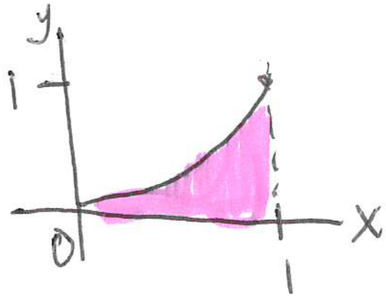
$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$\text{where } c_i = a + \Delta x\, i$$

$$\Delta x = \frac{b-a}{n}$$

ex Find the area of the region bounded by $f(x) = x^2$ and the x-axis over the interval $[\underset{a}{0}, \underset{b}{1}]$.

Step 1 Graph $f(x)$



Step 2 Find $\Delta x = \frac{b-a}{n}$

$$\Delta x = \frac{1-0}{n} = \boxed{\frac{1}{n}}$$

Step 3 Find $c_i = a + \Delta x i$

$$c_i = 0 + \frac{1}{n} i$$

$$= \boxed{\frac{1}{n} i}$$

Step 4 Find area using limits

$f(x) = x^2$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

plug in parts

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n} i\right)^2 \left(\frac{1}{n}\right)$$

simplify

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n^2} i^2\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3} i^2$$

move constant outside

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)$$

use properties of summation

$$= \lim_{n\to\infty} \left(\frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$$

plug in ∞

$$= \frac{1}{3} + \frac{1}{\infty} + \frac{1}{\infty}$$

$$= \frac{1}{3} + 0 + 0$$

$$= \frac{1}{3}$$

Summary