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Key

AP Calculus AB - Definite Integral as the limit of a Riemann Sum

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Learning objective: Interpret the definite integral as the limit of a Riemann sum; interpret the limit of a Riemann sum as a definite integral.

$\boxed{\lim \sum_{i=1}^{n} f(c_i)\Delta x}$

Main calculus concept/formula: $\lim\limits_{n\to\infty}\sum_{k=1}^{n} f(a+k\Delta x)\Delta x = \int_a^b f(x)\,dx$

$\boxed{\Delta x = \frac{b-a}{n} \quad c_i = a + \Delta x_i}$

Part I: Translate the definite integral into a Riemann sum. Do NOT evaluate.

Example: $\int_0^3 (x^2-1)\,dx = \lim\limits_{n\to\infty}\sum_{k=1}^{n}\left[\left(\left(\frac{3k}{n}\right)^2-1\right)\left(\frac{3}{n}\right)\right]$

$\Delta x = \frac{3-0}{n} = \boxed{\frac{3}{n}}$

$c_i = 0 + \frac{3}{n}i = \boxed{\frac{3}{n}i}$

1) $\int_1^2 (x+1)\,dx = \lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{1}{n}i\right)+1\right]\left(\frac{1}{n}\right)$

$\Delta x = \frac{2-1}{n} = \boxed{\frac{1}{n}}$

$c_i = \boxed{1+\frac{1}{n}i}$

2) $\int_2^4 (3x^2-1)\,dx$

$\Delta x = \frac{4-2}{n} = \frac{2}{n}$

$c_i = 2 + \frac{2}{n}i$

$\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[3\left(2+\frac{2}{n}i\right)^2-1\right]\left(\frac{2}{n}\right)$

3) $\int_0^{\pi} (\sin x)\,dx$

$\Delta x = \frac{\pi-0}{n} = \frac{\pi}{n}$

$c_i = 0 + \frac{\pi}{n}i = \frac{\pi}{n}i$

$\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[\sin\left(\frac{\pi}{n}i\right)\right]\left(\frac{\pi}{n}\right)$

4) $\int_{-2}^{1} (3x^2+2)\,dx$

$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}$

$c_i = -2 + \frac{3}{n}i$

$\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[3\left(-2+\frac{3}{n}i\right)^2+2\right]\left(\frac{3}{n}\right)$

5) $\int_3^7 (2x^2+3x)\,dx$

$\Delta x = \frac{7-3}{n} = \frac{4}{n}$

$c_i = 3 + \frac{4}{n}i$

$\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[2\left(3+\frac{4}{n}i\right)^2+3\left(3+\frac{4}{n}i\right)\right]\left(\frac{4}{n}\right)$

1) Using the right and left approximation with 6 equal subdivisions each, estimate the distance traveled over [0, 3] if the velocity of an object at half-second intervals is:

Riemann Sum

$\sum_{i=1}^{n} f(c_i)\Delta x \quad \Delta x = \frac{b-a}{n}$

t	0	0.5	1	1.5	2	2.5	3
v (ft/sec)	0	12	18	20	14	20	20

$n = 6 \quad \Delta x = \frac{3-0}{6} = \frac{1}{2}$

$[0, 3]$ a b

left approx $= \Delta x \sum_{i=1}^{n} f(c_i)$

$= \frac{1}{2}[0 + 0.5 + 1 + 1.5 + 2 + 2.5]$

$= \boxed{42 \text{ feet}}$

right approx $= \frac{1}{2}[12+18+20+14+20+20]$

$= 52$ feet

2) Find the distance of the object over the interval [0,3] in the previous problem using 3 subdivisions with a midpoint approximation.

$[0, 3]$ a b $\quad \Delta x = \frac{3-0}{3} = 1$

$n = 3$

t	0	.5	1
v	0	12	18

$x_1 = \frac{1+0}{2} = 0.5$

1	1.5	2
18	20	14

$x_2 = \frac{1+2}{2} = 1.5$

2	2.5	3
14	20	20

$x_3 = \frac{2+3}{2} = 2.5$

Midpoint Rule $= \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$

midpt approx $= 1[f(.5) + f(1.5) + f(2.5)] = 1[12 + 20 + 20]$

$= \boxed{52 \text{ feet}}$

3) Find the distance of the object over the interval [0, 3] in the problem 1 using 6 subdivisions with a trapezoidal approximation.

$[0, 3]$ a b

$n = 6$

Trapezoidal Rule $= \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]$

$= \frac{3-0}{2(6)}[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)]$

$= \frac{1}{4}[0 + 2(12) + 2(18) + 2(20) + 2(14) + 2(20) + 20] = \boxed{47 \text{ feet}}$

4) A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate R(t) on October 21 is recorded, in inches per hour, in the following table, where t is the number of hours since midnight. Compute the total rainfall during this 24-hour period using a right and a left approximation with 5 subdivisions.

notice that the values of t are not equal intervals: Δx is not constant

t	0	3	5	10	16	24
R(t)	0.2	0.1	0.4	1.0	0.6	0.25

left approx. $= (3-0)f(0) + (5-3)f(3) + (10-5)f(5) + (16-10)f(10) + (24-16)f(16)$

$= 3(0.2) + 2(0.1) + 5(0.4) + 6(1) + 8(0.6)$

$= 13.6$ inches

Right approx. $= 3f(3) + 2f(5) + 5f(10) + 6f(16) + 8f(24) = 3(0.1) + 2(0.4) + 5(1.0) + 10(0.6) + 16(0.25) = 11.7$ inches

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

1a) $v'(16) = \dfrac{V(20) - V(12)}{20 - 12}$

$= \dfrac{240 - 200}{8}$

$= \dfrac{40}{8}$

$\boxed{v'(16) = 5\ \text{m/min}^2}$

(b) The meaning of $\int_0^{40} |v(t)|\,dt$ represents the total distance Johanna jogs in meters over the time period of 0 minutes to 40 minutes.

Since the values of t are not equal intervals, Δx is not constant

$\int_0^{40} |v(t)|\,dt \approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)|$

$= 12(200) + 8(240) + 4(220) + 16(150)$

$= \boxed{7600 \text{ meters}}$

c) Since $B(t)$ = Velocity then $B'(t)$ = acceleration

$B'(t) = 3t^2 - 12t$

$B'(5) = 3(5)^2 - 12(5)$

$= 15\ \text{m/min}^2$

$\boxed{\text{Bob's acceleration at } t = 5 \text{ min is } 15\ \text{m/min}^2}$

d) Avg Vel. $= \dfrac{1}{b-a}\int_a^b f(x)\,dx$

$= \dfrac{1}{10-0}\int_0^{10} B(t)\,dt$

$= \dfrac{1}{10}\int_0^{10} (t^3 - 6t^2 + 300)\,dt$

$= \dfrac{1}{10}\left[\dfrac{t^4}{4} - \dfrac{6t^3}{3} + 300t\right]_0^{10}$

$= \dfrac{1}{10}\left[\dfrac{10{,}000}{4} - 2000 + 3000\right]$

$= \boxed{350\ \text{m/min}}$

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

notice that t values are not equal Intervals

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

    (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

    (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

    (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

a) $R'(2) = \dfrac{R(3)-R(1)}{3-1}$

$= \dfrac{950-1190}{2}$

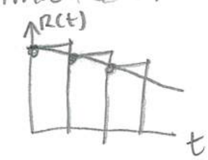
$= \boxed{-120 \text{ Liters/hour}^2}$

b) Left approx $\approx (1-0)f(0)+(3-1)f(1)+(6-3)f(3)+(8-6)f(6)$

$= 1(1340)+2(1190)+3(950)+2(740)$

$=$ 8050 liters of water removed

Since $R(t)$ is decreasing, a left Riemann approximation is an overestimate of the total amount of water removed.



c) Since at $t=0$, there is 50,000 Liters of water in the tank and $W(t)$ is the function of water being pumped, we subtract 8050 Liters of water removed.

total water in the water $\approx 50{,}000 + \int_0^8 W(t)\,dt - 8050$

$\approx 50{,}000 + 7836.195 - 8050$

$\approx \boxed{49{,}786 \text{ liters}}$

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

notice t-values are not equal interval, thus height is not constant.

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

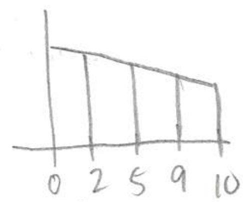
   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

a) $H'(3.5) = \dfrac{H(5)-H(2)}{5-2}$

$= \dfrac{52-60}{3}$

$= \boxed{\dfrac{-8}{3}\ \text{Celsius/minutes}}$



b) $\frac{1}{10}\int_0^{10} H(t)dt$ represents the average temperature in the pot in Celsius, from $0 \le t \le 10$ minutes. Since $A = \frac{1}{2}(b_1+b_2)h = \frac{h}{2}(b_1+b_2)$

$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left[\left(\frac{2-0}{2}\right)(f(0)+f(2)) + \left(\frac{5-2}{2}\right)(f(2)+f(5)) + \left(\frac{9-5}{2}\right)(f(5)+f(9)) + \left(\frac{10-9}{2}\right)(f(9)+f(10))\right]$

$= \frac{1}{10}\left[1(66+60) + \frac{3}{2}(60+52) + 2(52+44) + \frac{1}{2}(44+43)\right]$

Avg temp $= \boxed{52.95}$ celsius

c) $\int_0^{10} H'(t)dt = [H(10) - H(0)]$

$= 43 - 66$

$= -23$ celsius

The temperature dropped 23° celsius from $0 \le t \le 10$ minutes.

notice Interval
notice constant

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

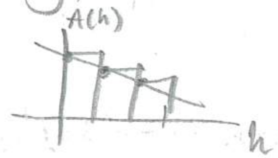
   (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

   (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

   (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

a) Left approx $\approx (2-0)f(0) + (5-2)f(2) + (10-5)f(5)$
$= 2(50.3) + 3(14.4) + 5(6.5)$
$= 176.3$ feet

b) Since $A(h)$ is decreasing, left approximation is an overestimation.



c) Since $A(h)$ is feet^2 going backwards gets feet^3 therefore

$\int_0^{10} A(h)\,dh \approx 101.325$ feet^3, $\boxed{\text{the volume of the tank is } 101.325 \text{ feet}^3}$

Explicit explanation of how to do (b).

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

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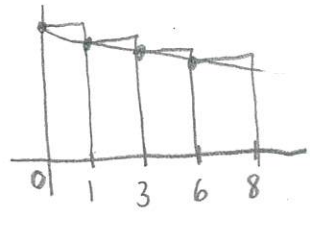
(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(a) $R'(2) = \frac{R(3)-R(1)}{3-1} = \frac{950-1190}{3-1} = -120$ liters/hr^2

b) Notice that the t-values are not in equal intervals this mean we dont divide by n because the rectangles does not have a constant base: $\Delta x = \frac{b-a}{n} \rightarrow$ is now $\boxed{\Delta x = b-a}$

left Riemann $= \sum_{i=1}^{n} f(c_i)\Delta x$



Δx $f(c_i)$

$\approx (1-0)f(0) + (3-1)f(1) + (6-3)f(3) + (8-6)f(6)$

why function notation? If do not use function notation, you will lose points. Function notation explains where you get the value from.

$= 1(1340) + 2(1190) + 3(950) + 2(740)$

$= \boxed{8050 \text{ liters of water removed}}$

c) Total $\approx 5000 + \int_0^8 W(t)\,dt - 8050$

$= 5000 + 7836.195 - 8050$

$= \boxed{149{,}786 \text{ liters}}$